Exercise 110

Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

Solution

Suppose that

$$f(g(x)) = x.$$

Differentiate both sides with respect to x.

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}(x)$$

Use the chain rule on the left side.

$$f'(g(x)) \cdot g'(x) = 1$$

Use the fact that $f'(x) = 1 + [f(x)]^2$.

$$\left\{1 + [f(g(x))]^2\right\} \cdot g'(x) = 1$$

Use the fact that $f(g(x)) = x$.

$$(1 + x^2) \cdot g'(x) = 1$$

Therefore, dividing both sides by $1 + x^2$,

$$g'(x) = \frac{1}{1 + x^2}.$$